JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **188**, 885–900 (1994)

The Weighted Logarithmic Mean

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Submitted by J. L. Brenner

Received August 23, 1993

1. INTRODUCTION AND NOTATION

The unweighted logarithmic mean of the positive numbers x and y is defined by

$$\begin{aligned} L(x, y) &= \frac{x-y}{\ln x - \ln y}, \qquad x \neq y \\ L(x, x) &= x \end{aligned} \tag{1.1}$$

(see, e.g., [4, p. 346; 7, Eq. (1); 8, Sect. 2.6]). The mean L plays a role in the analysis of heat transmission between two fluids (see [1]). Another application is in the study of distribution of electrical charge on a conductor (cf. [24]). It follows from (1.1) that $L(x, y)$ is symmetric in x and y and homogeneous of order one in its variables. An integral formula for L

$$L(x, y) = \left[\int_0^1 \frac{dt}{tx + (1-t)y}\right]^{-1} \tag{1.2}$$

appears in Carlson [7, Eq. (3)].

Although (1.1) does not give insights about possible generalizations to several variables, the integral (1.2) has been employed to achieve this goal. Following [5, Eq. (2.4)] the unweighted logarithmic mean of the positive numbers $x_1, \ldots, x_n$ is defined by

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0022-247X/94 $6.00

$$L(x_1, \ldots, x_n) = \left[(n-1)! \int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i x_i\right)^{-1} du\right]^{-1}, \tag{1.3}$$

where

$$E_{n-1} = \{(u_1, \ldots, u_{n-1}) : u_i \geq 0, 1 \leq i \leq n-1, u_1 + \cdots + u_{n-1} \leq 1\}$$

is the Euclidean simplex, $u_n = 1 - u_1 - \cdots - u_{n-1}$, and $du = du_1 \cdots du_{n-1}$.

Further generalization of the logarithmic mean of several variables is discussed in [3]. Let μ be a probability measure on E_{n-1} and let

$$X = (x_1, \ldots, x_n) \in \mathbb{R}^n_>,$$

where $\mathbb{R}_>$ stands for the set of all positive real numbers. The weighted logarithmic mean $L(\mu; X)$ is defined by

$$L(\mu; X) = \left[\int_{E_{n-1}} \left(\sum_{i=1}^{n} u_i x_i\right)^{-1} d\mu(u)\right]^{-1} \tag{1.4}$$

(see [3, Eq. (3.6)]). The new mean is homogenous of order one. In contrast to the unweighted mean (1.3), $L(\mu; X)$ is no longer symmetric in its variables for a general measure μ. For particular probability measures the symmetry of the weighted logarithmic mean can be restored easily. An important example of such a measure is the Dirichlet measure μ_b. For $b = (b_1, \ldots, b_n) \in R^n_>$, $n \geq 2$, it is defined by

$$\mu_b(u) = \prod_{i=1}^{n} u_i^{b_i - 1}/B(b) \tag{1.5}$$

(see, e.g., [8, Eq. (4.4-1)]). Here B stands for the multivariate beta function.

This paper deals with a new weighted logarithmic mean of several variables. Throughout the sequel the new mean will be denoted by $\mathcal{L}$ rather than L. In Section 2 we give the definition of $\mathcal{L}$. Basic properties and inequalities obeyed by $\mathcal{L}$ are also included. The next section of this paper is devoted to the discussion of inequalities in the bivariate case. The main tools employed are the elementary quadrature formulas. In Section 4 we deal with the logarithmic mean $\mathcal{L}(\mu_b; X)$. It is shown that the mean in question can be regarded as a confluent hypergeometric mean of several variables. New integrals, a limit theorem, and inequalities are also included.

2. The New Weighted Logarithmic Mean

We begin with another integral formula for the logarithmic mean $L(x, y)$, namely,

$$L(x, y) = \int_0^1 x^t y^{1-t}\, dt. \tag{2.1}$$

A verification of (2.1) is an elementary task in calculus. Formula (2.1) suggests a generalization to several variables. For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n_>$, $n \geq 2$, and a probability measure μ on E_{n-1}, we define

$$\mathcal{L}(\mu; X) = \int_{E_{n-1}} \prod_{i=1}^{n} x_i^{u_i}\, d\mu(u), \tag{2.2}$$

where $u = (u_1, \ldots, u_{n-1}) \in E_{n-1}$ and $u_n = 1 - u_1 - \cdots - u_{n-1}$. Use of $x^\alpha = \exp(\alpha \ln x)$ puts (2.2) in another form

$$\mathcal{L}(\mu; X) = \int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i \ln x_i\right) d\mu(u). \tag{2.3}$$

Some elementary properties of $\mathcal{L}$ follow immediately from (2.2)–(2.3). We have

$$\min\{x_i\colon 1 \leq i \leq n\} \leq \mathcal{L}(\mu; X) \leq \max\{x_i\colon 1 \leq i \leq n\}, \tag{2.4}$$

$$\mathcal{L}(\mu; x, \ldots, x) = x, \qquad x > 0 \tag{2.5}$$

$$\mathcal{L}(\mu; \alpha X) = \alpha \mathcal{L}(\mu; X), \qquad \alpha > 0, \tag{2.6}$$

where $\alpha X := (\alpha x_1, \ldots, \alpha x_n)$,

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \mathcal{L}(\mu; X) = \mathcal{L}(\mu; X). \tag{2.7}$$

The last formula is Euler's equation for a homogeneous function with order of homogeneity equal to one.

Another integral formula for $\mathcal{L}$ can be derived with the aid of the generalized univariate B-splines. For the reader's convenience we recall the definition of this class of splines. For $Z = (z_1, \ldots, z_n) \in \mathbb{R}^n$ with

$$\min\{z_i\colon 1 \leq i \leq n\} < \max\{z_i\colon 1 \leq i \leq n\}$$

the generalized B-spline $B(\cdot|\mu; Z)$ can be realized as the kernel of the distribution

$$\int_{E_{n-1}} f\left(\sum_{i=1}^{n} u_i z_i\right) d\mu(u), \qquad f \in C_0^\infty(\mathbb{R})$$

(see [11]). Letting $f(t) = \exp t$, $z_i = \ln x_i$, $1 \le i \le n$, we obtain with the aid of (2.3)

$$\mathcal{L}(\mu; X) = \int_\xi^\eta e^t B(t|\mu; \ln X)\, dt, \tag{2.8}$$

where $\ln X := (\ln x_1, \ldots, \ln x_n)$ and ξ and η denote the smallest and the largest elements of $\ln X$, respectively. It follows immediately from (2.8) that

$$\mathcal{L}(\mu; X) = \sum_{m=0}^{\infty} \nu_m/m!,$$

where

$$\nu_m = \int_\xi^\eta t^m B(t|\mu; \ln X)\, dt, \qquad m \in \mathbb{N}$$

is the mth order moment of B.

If μ is the uniform measure on E_{n-1}, i.e., $\mu = (n-1)!$, then the corresponding spline becomes the B-spline of Curry and Schoenberg [10]. Moments of the latter class are discussed in [16–17]. It is worth mentioning that the complete symmetric functions, the q-binomial coefficients (Gauss polynomials), and the extended r-Stirling numbers of the second kind all can be represented in terms of the moments of the B-splines. When $\mu = \mu_b$, then the corresponding splines are called the Dirichlet splines. Their moments are discussed in [20].

We now give a probabilistic interpretation of $\mathcal{L}(\mu; X)$. Let $U = (U_1, \ldots, U_n)$ be a random vector with the joint probability density function μ over E_{n-1}. Further, let

$$h(U) = \exp(U_1 \ln x_1 + \cdots + U_n \ln x_n)$$

($x_i > 0$, $1 \le i \le n$) be a random variable. Then the expected value of h is given by

$$E(h(U)) = \mathcal{L}(\mu; X).$$

Before we state the main result of this section let us introduce more notation. By w_i, where

$$w_i = \int_{E_{n-1}} u_i \, d\mu(u) \tag{2.9}$$

$(1 \le i \le n)$ we will denote the ith weight associated with the probability measure μ on E_{n-1}. Clearly $w_i > 0$, $1 \le i \le n$, and $w_1 + \cdots + w_n = 1$. For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n_>$, $Y = (y_1, \ldots, y_n) \in \mathbb{R}^n_>$, and $\alpha, \beta \in \mathbb{R}$ we define

$$X^\alpha Y^\beta = (x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) \qquad \text{and}$$

$$\alpha X + \beta Y = (\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n).$$

We are in a position to prove the main result of this section.

THEOREM 1. *Let μ be a probability measure on E_{n-1} and let the weights $w_1, \ldots, w_n$ be the same as in (2.9). If $x_1, \ldots, x_n$ are not all equal, then*

$$\prod_{i=1}^n x_i^{w_i} < \mathscr{L}(\mu; X) < \sum_{i=1}^n w_i x_i. \tag{2.10}$$

If $X \neq Y$, then for $\alpha, \beta \in \mathbb{R}_>$ with $\alpha + \beta = 1$

$$\mathscr{L}(\mu; X^\alpha Y^\beta) < \mathscr{L}(\mu; \alpha X + \beta Y) \tag{2.11}$$

and

$$\mathscr{L}(\mu; X^\alpha Y^\beta) < \mathscr{L}(\mu; X)^\alpha \mathscr{L}(\mu; Y)^\beta. \tag{2.12}$$

Proof. We shall establish inequality (2.10) using the following one

$$f\left(\sum_{i=1}^n w_i z_i\right) \le \int_{E_{n-1}} f\left(\sum_{i=1}^n u_i z_i\right) d\mu(u) \le \sum_{i=1}^n w_i f(z_i), \tag{2.13}$$

where f is a convex function on a closed interval $[c, d]$, $z_1, \ldots, z_n \in [c, d]$. The inequalities in (2.13) are strict provided f is not a polynomial of degree one or less and they are reversed if f is concave on $[c, d]$. This result has been established in [18, Theorem 4.2]. Letting $f(t) = \exp t$, $z_i = \ln x_i$, $1 \le i \le n$, in (2.13) we obtain the desired result utilizing the representation (2.3). Inequality (2.11) can be regarded as a generalization of the inequality for the weighted arithmetic and geometric means

$$x^\alpha y^\beta \le \alpha x + \beta y, \qquad x > 0, y > 0.$$

This in turn implies

$$\prod_{i=1}^{n} (x_i^{\alpha} y_i^{\beta})^{u_i} < \prod_{i=1}^{n} (\alpha x_i + \beta y_i)^{u_i}, \qquad u_i > 0.$$

Integration over E_{n-1} with respect to the measure μ yields the assertion. We shall now establish (2.12). For the sake of brevity let $t_i = \ln x_i$, $v_i = \ln y_i$, $1 \le i \le n$. Using (2.3), Hölder's inequality, and (2.3) again, we obtain successively

$$\begin{aligned}
\mathcal{L}(\mu; X^{\alpha} Y^{\beta}) &= \int_{E_{n-1}} \exp\left[\sum_{i=1}^{n} u_i(\alpha t_i + \beta v_i)\right] d\mu(u) \\
&= \int_{E_{n-1}} \left[\exp\left(\sum_{i=1}^{n} u_i t_i\right)\right]^{\alpha} \left[\exp\left(\sum_{i=1}^{n} u_i v_i\right)\right]^{\beta} d\mu(u) \\
&< \left[\int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i t_i\right) d\mu(u)\right]^{\alpha} \left[\int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i v_i\right) d\mu(u)\right]^{\beta} \\
&= \mathcal{L}(\mu; X)^{\alpha} \mathcal{L}(\mu; Y)^{\beta}. \quad \blacksquare
\end{aligned}$$

3. Inequalities for the Logarithmic Mean of Two Variables

The purpose of this section is to establish some inequalities for the weighted and unweighted logarithmic mean of two variables. The key tools employed here are some elementary quadrature formulas. This approach has one advantage over other methods used previously. A knowledge of the error terms allows us to estimate the sharpness of the resulting inequalities.

To this end, let x and y be two distinct positive numbers. Further, let $A(x, y) \equiv A = (x + y)/2$ and let $G(x, y) \equiv G = (xy)^{1/2}$ be the unweighted arithmetic and geometric means of x and y, respectively. It is well known that the unweighted logarithmic mean $L(x, y)$ interpolates the arithmetic and geometric means, i.e.,

$$G(x, y) < L(x, y) < A(x, y) \tag{3.1}$$

(see [7–8, 22]). Replacing x by $x^{1/2}$ and y by $y^{1/2}$ Carlson [7] obtained a refinement of (3.1), namely,

$$G(x, y) < (xy)^{1/4} \frac{x^{1/2} + y^{1/2}}{2} < L(x, y) < \left[\frac{x^{1/2} + y^{1/2}}{2}\right]^2 < A(x, y). \tag{3.2}$$

In this section we give, among other things, generalizations of (3.1) and (3.2) to the weighted logarithmic mean $\mathcal{L}(\mu; x, y)$. It follows from (2.2) that

$$\mathcal{L}(\mu; x, y) = \int_0^1 x^t y^{1-t} \mu(t)\, dt, \tag{3.3}$$

where now μ is the probability measure on the unit interval. The first quadrature formula we shall apply to (3.3) is the composite midpoint rule. If f is twice continuously differentiable on [0, 1] and if $t_j = (2j - 1)/2m$, $1 \le j \le m$, then

$$\int_0^1 f(t)\, dt = \frac{1}{m} \sum_{j=1}^m f(t_j) + \frac{1}{24m^2} f''(\xi_m), \qquad 0 < \xi_m < 1$$

(see, e.g., [14, Chap. 7, Sect. 1.1]). Letting $f(t) = x^t y^{1-t} \mu(t)$ and assuming that f is twice continuously differentiable on [0, 1], we obtain

$$\lambda/24m^2 < \mathcal{L}(\mu; x, y) - \frac{1}{m} \sum_{j=1}^m x^{t_j} y^{1-t_j} \mu(t_j) < \Lambda/24m^2, \tag{3.4}$$

provided f is convex on [0, 1]. Here

$$\lambda = \min\{f''(t)\colon 0 \le t \le 1\}, \qquad \Lambda = \max\{f''(t)\colon 0 \le t \le 1\}.$$

If $\mu = \mu_b$ – the Dirichlet measure on E_1 and $b = (b_1, b_2)$ with $0 < b_1$, $b_2 \le 1$, then $x^t y^{1-t} \mu_b(t)$, $0 < t < 1$, is strictly log-convex in the stated domain. Writing $\mathcal{L}(b; x, y)$ for $\mathcal{L}(\mu_b; x, y)$ we obtain with the aid of (3.4)

$$\lambda/24m^2 < \mathcal{L}(b; x, y) - \frac{1}{m} \sum_{j=1}^m x^{t_j} y^{1-t_j} \mu_b(t_j) < \Lambda/24m^2. \tag{3.5}$$

In particular, if $b = (1, 1)$ and $m = 1$, then (3.5) provides

$$r[\ln(x/y)]^2/24 < L(x, y) - G(x, y) < s[\ln(x/y)]^2/24, \tag{3.6}$$

where $r = \min\{x, y\}$, $s = \max\{x, y\}$. Similarly, if $m = 2$, then (3.5) gives

$$r[\ln(x/y)]^2/96 < L(x, y) - (xy)^{1/4} \frac{x^{1/2} + y^{1/2}}{2} < s[\ln(x/y)]^2/96. \tag{3.7}$$

Thus (3.6) and (3.7) provide information about the sharpness of the first inequality in (3.1) and the second inequality in (3.2), respectively.

To obtain more inequalities for the mean under discussion we shall use the composite trapezoidal rule. If $f \in C^2[0, 1]$ and if $v_i = i/m, 0 \leq i \leq m$, then

$$\int_0^1 f(t)\, dt = \frac{1}{m} \sum_{i=0}^{m}{}'' f(v_i) - \frac{1}{12m^2} f''(\eta_m), \qquad 0 < \eta_m < 1$$

(cf. [14, Chap. 7, Sect. 1.2]). Here the symbol Σ'' means that the first and last terms are halved. Application to (3.3) provides

$$\lambda/12m^2 < \frac{1}{m} \sum_{i=0}^{m}{}'' x^{v_i} y^{1-v_i} \mu(v_i) - \mathscr{L}(\mu; x, y) < \Lambda/12m^2 \tag{3.8}$$

provided the function $x^t y^{1-t} \mu(t)$ has a positive second order derivative on $[0, 1]$. If $\mu(t) = 1$, then the second inequality in (3.1) and the third inequality in (3.2) follow from (3.8) by letting $m = 1$ and $m = 2$, respectively.

Lin [15, Theorem 1] has proved that

$$L(x, y) < M_{1/3}(x, y),$$

where

$$M_{1/3}(x, y) = \left(\frac{x^{1/3} + y^{1/3}}{2}\right)^3$$

is the power mean of order 1/3. Clearly the last inequality is stronger than the third inequality in (3.2). We shall show that the sharpness of Lin's inequality is measured by

$$r[\ln(x/y)]^4/6480 < M_{1/3}(x, y) - L(x, y) < s[\ln(x/y)]^4/6480.$$

For the proof we apply the three-eighths rule [2, Tab. 5.8]

$$\int_0^1 f(t)\, dt = \frac{1}{8}\left[f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right] - \frac{1}{6480} f^{(4)}(\eta),$$

$0 < \eta < 1$, to (2.1) to obtain

$$L(x, y) = M_{1/3}(x, y) - [\ln(x/y)]^4 x^\eta y^{1-\eta}/6480.$$

Hence the assertion follows.

Another source of inequalities for the logarithmic mean of two variables is the composite Simpson rule [14, Chap. 7, Sect. 1.4]

$$\int_0^1 f(t)\,dt = \frac{1}{3m}\left[f(v_0) + 4\sum_{i=1,3,\ldots,m-1} f(v_i) + 2\sum_{i=2,4,\ldots,m-2} f(v_i) + f(v_m)\right] - \frac{1}{180m^4}f^{(4)}(\varepsilon_m),$$

where $f \in C^4[0, 1]$, $0 < \varepsilon_m < 1$, $v_i = i/m$, $0 \le i \le m$, m-even. Use of Simpson's rule on (2.1) provides

$$r[\ln(x/y)]^4/180m^4 < 2\left[\sum_{i=0}^{m}{}'' x^{v_i}y^{1-v_i} + \sum_{i=1,3,\ldots,m-1} x^{v_i}y^{1-v_i}\right]\Big/3m - L(x, y) < s[\ln(x/y)]^4/180m^4.$$

For $m = 2$ this simplifies to

$$r[\ln(x/y)]^4/2880 < [A(x, y) + 2G(x, y)]/3 - L(x, y) < s[\ln(x/y)]^4/2880.$$

We close this section with the following inequality

$$[x^\rho y^{1-\rho}\mu(\rho) + x^{1-\rho}y^\rho\mu(1-\rho)]/2 < \mathscr{L}(\mu; x, y). \tag{3.9}$$

The last inequality is valid for any $1/4 \le \rho \le 1/2$ and any probability measure μ for which

$$(d^2/dt^2)[x^t y^{1-t}\mu(t)] > 0, \qquad \text{for } 0 \le t \le 1.$$

This follows immediately from (3.3) and from

$$\int_0^1 f(t)\,dt = [f(\rho) + f(1-\rho)]/2 + f''(\xi)p(\rho), \tag{3.10}$$

$0 < \xi < 1$, where $p(\rho) = (\rho - \rho^2 - 1/6)/2$ and f is assumed to be twice continuously differentiable on the unit interval. In order to establish (3.10) let us consider a quadrature formula

$$\int_0^1 f(t)\,dt = c[f(v_1) + f(v_2)] + \Psi f,$$

$v_1, v_2 \in (0, 1)$. We shall determine c, v_1, v_2, and Ψf by requiring that the remainder Ψf vanishes for $f(t) = 1, t$. This implies that $c = 1/2$ and $v_1 + v_2 = 1$. Writing ρ for v_1 we have

$$\int_0^1 f(t)\,dt = [f(\rho) + f(1-\rho)]/2 + \Psi f, \tag{3.11}$$

$0 < \rho \le 1/2$.

Our goal is to establish a closed formula for Ψf. To this aim we employ Peano's kernel theorem [14, p. 192] to obtain

$$\Psi f = \int_0^1 f''(z) K_1(z)\,dz, \tag{3.12}$$

where

$$K_1(z) = \Psi(\cdot - z)_+ \tag{3.13}$$

and

$$(t-z)_+ = \begin{cases} t - z & \text{if } t \ge z \\ 0 & \text{otherwise.} \end{cases}$$

Combining (3.13) with (3.11) yields

$$K_1(z) = \int_0^1 (t-z)_+\,dt - [(\rho - z)_+ + (1 - \rho - z)_+]/2.$$

A simple computation shows that

$$K_1(z) = \begin{cases} z^2/2, & 0 \le z \le \rho, \\ (z^2 - z + \rho)/2, & \rho \le z \le 1 - \rho, \\ (1-z)^2/2, & 1 - \rho \le z \le 1. \end{cases}$$

Hence $K_1(z) \ge 0$, $0 \le z \le 1$, provided $\rho \ge 1/4$. This in conjunction with (3.12) gives

$$\Psi = f''(\xi) p(\rho).$$

Here we have used the Mean Value Theorem for integrals. Combining this with (3.11) yields the assertion. Pittenger [21, Theorem 1] has proven that for the unweighted logarithmic mean the inequality (3.9) is valid on a slightly larger domain for ρ, namely,

$$1/2 \ge \rho \ge (1 - (3)^{-1/2})/2 = 0.211 \cdots.$$

4. The Mean $\mathscr{L}(b; X)$

This section deals with the logarithmic mean which is generated by the Dirichlet measure μ_b (see (1.5)). Throughout the sequel we will write $\mathscr{L}(b; X)$ for $\mathscr{L}(\mu_b; X)$. We shall show that the mean under discussion is the confluent hypergeometric function S. For $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n_>$ this function has an integral representation

$$S(b; Z) = \int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i z_i\right) d\mu_b(u) \tag{4.1}$$

(see [8, Eq. (5.8-1)]). Comparison with (2.3) yields

$$\mathscr{L}(b; X) = S(b; \ln X), \tag{4.2}$$

where the set $\ln X$ is the same as in (2.8). We list below some properties which are peculiar for the mean $\mathscr{L}(b; X)$:

(i) $\mathscr{L}(b_1, \ldots, b_n; x_1, \ldots, x_n)$ is symmetric in the indices $1, \ldots, n$ [8, Theorem 5.2-3].

(ii) A vanishing parameter b_i can be omitted along with the corresponding variable x_i [8, Eq. (6.3-3)].

(iii) Equal variables can be replaced by a single variable if the corresponding parameters are replaced by their sum [8, Eq. (5.2-3)].

We now give examples which illuminate a connection of $\mathscr{L}(b; X)$ with confluent hypergeometric functions. Use of [8, Eq. (5.8-6)] on (4.2) gives

$$\mathscr{L}(b_1, b_2; x_1, x_2) = x_2\,[{}_1F_1(b_1; b_1 + b_2; \ln(x_1/x_2))],$$

where ${}_1F_1$ is Kummer's confluent hypergeometric function. Similarly, application of [19, Eq. (6.3.1)] to (4.2) provides

$$\mathscr{L}(b_1, b_2, b_3; x_1, x_2, x_3) = x_3[\Phi_2(b_1, b_2; b_1 + b_2 + b_3; \ln(x_1/x_3), \ln(x_2/x_3))],$$

where Φ_2 stands for the Humbert confluent hypergeometric function (see [13, Eq. 5.7(21)]).

Our next goal is to obtain some integrals for the mean in question. Assume that the vectors $b = (b_1, \ldots, b_n)$, $d = (d_1, \ldots, d_m)$, $X = (x_1, \ldots, x_n)$, and $Y = (y_1, \ldots, y_m)$, $m, n > 1$, are positive. Also, let

$$\beta = b_1 + \cdots + b_n, \qquad \delta = d_1 + \cdots + d_m. \tag{4.3}$$

By $\mu_{(\beta,\delta)}$, β, $\delta > 0$, we will denote the Dirichlet measure on E_1. Then

$$\mathscr{L}(b, d; X, Y) = \int_0^1 \mathscr{L}(b; X^t)\mathscr{L}(d; Y^{1-t})\, d\mu_{(\beta,\delta)}(t), \tag{4.4}$$

where

$$\mathscr{L}(b, d; X, Y) = \mathscr{L}(b_1, \ldots, b_n, d_1, \ldots, d_m; x_1, \ldots, x_n, y_1, \ldots, y_m).$$

This follows easily by use of [19, Theorem 3.1] on (4.2). For later reference let

$$a = (b_1, \ldots, b_{n-1}), \qquad \beta_n = b_1 + \cdots + b_{n-1}. \tag{4.5}$$

Application of [19, Corollary 3.2] to (4.2) provides

$$\mathscr{L}(b; X) = \int_0^1 \mathscr{L}(a; Z)\, d\mu_{(\beta_n, b_n)}(t), \tag{4.6}$$

where $Z = (z_1, \ldots, z_{n-1})$ with $z_i = x_i^t x_n^{1-t}$, $1 \le i \le n - 1$.

Finally, if $\lambda > 0$, then

$$\mathscr{L}(b_1, \ldots, b_{n-1}, b_n + \lambda; X) = \int_0^1 \mathscr{L}(b; W)\, d\mu_{(\beta,\lambda)}(t), \tag{4.7}$$

where β is defined in (4.3) and

$$W = x_n((x_1/x_n)^t, \ldots, (x_{n-1}/x_n)^t, 1).$$

This last integral is a special case of [19, Corollary 3.4].

Before we state the next result, let us recall the definition of the R-hypergeometric polynomial of several variables. For $d = (d_1, \ldots, d_n) \in \mathbb{C}^n$ and $Z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ the mth order R-hypergeometric polynomial R_m is defined by [8, Eq. (6.2-1)]

$$R_m(d; Z) = \frac{m!}{(\delta, m)} \sum \frac{(d_1, i_1) \cdots (d_n, i_n)}{i_1! \cdots i_n!} z_1^{i_1} \cdots z_n^{i_n}, \tag{4.8}$$

where δ is defined in (4.3), $(\delta, m) = \Gamma(\delta + m)/\Gamma(\delta)$, $\delta \ne 0, -1, \ldots,$ is the Appell symbol. In (4.8) the summation extends over all nonnegative integers $i_1, \ldots, i_n$ whose sum is m.

We are in a position to prove the following.

THEOREM 2. *Let $c, w_1, \ldots, w_n$ be strictly positive real numbers, assume that $w_1 + \cdots + w_n = 1$, and define $cw = (cw_1, \ldots, cw_n)$. Then*

$$\lim_{c \to 0} \mathscr{L}(cw; X) = \sum_{i=1}^{n} w_i x_i \tag{4.9}$$

and

$$\lim_{c \to \infty} \mathscr{L}(cw; X) = \prod_{i=1}^{n} x_i^{w_i}. \tag{4.10}$$

Proof. In order to establish the limit relation (4.9) we substitute $b = cw$ into (4.2) and next apply [8, Ex. 6.3–4]. For the proof of (4.10) we use [8, Eq. (6.3–5)] on (4.2) to obtain

$$\mathscr{L}(cw; X) = \sum_{m=0}^{\infty} \frac{1}{m!} R_m(cw; \ln X). \tag{4.11}$$

We shall prove first that the series in (4.11) converges uniformly in $0 < c < \infty$. This in turn implies further that as $c \to \infty$, we can proceed to the limit term by term. Making use of [8, (6.2–24)] we obtain

$$|R_m(cw; \ln X)| \leq |z|^m, \qquad m \in \mathbb{N},$$

where $|z| = \max\{|\ln x_i| : 1 \leq i \leq n\}$. By the Weierstrass M test the series in (4.11) converges uniformly in the stated domain. To complete the proof of (4.10) we use [8, Eq. (6.2-18)] to obtain

$$\lim_{c \to \infty} R_m(cw; \ln X) = \left(\ln \prod_{i=1}^{n} x_i^{w_i} \right)^m. \tag{4.12}$$

Application of (4.12) to (4.11) yields (4.10). ∎

We will now deal with inequalities for $\mathscr{L}(b; X)$. Replacing μ by μ_b in (2.13) and writing $F(b; Z)$ for

$$\int_{E_{n-1}} f\left(\sum_{i=1}^{n} u_i z_i \right) d\mu_b(u)$$

we have [8, Ex. 5.2–1]

$$f\left(\sum_{i=1}^{n} w_i z_i \right) \leq F(b; Z) \leq \sum_{i=1}^{n} w_i f(z_i), \tag{4.13}$$

where now $w_i = b_i/\beta$, $1 \leq i \leq n$ (cf. [8, (4.4-8)]). For later use we define

$$v = \beta_n/\beta, \qquad \rho_l = b_l/\beta_n, \tag{4.14}$$

$1 \leq l \leq n-1$, where β and β_n are given in (4.3) and (4.5), respectively. Also, let

$$Y = (y_i) \text{ with } y_i = vz_i + (1-v)z_n, \qquad 1 \leq i \leq n-1 \tag{4.15}$$

$$Z_{n-1} = (z_1, \ldots, z_{n-1}). \tag{4.16}$$

Two refinements of (4.13) are contained in

THEOREM 3. *Let f be convex on a closed interval containing $z_1, \ldots, z_n$. Then for $b \in \mathbb{R}^n_>$*

$$F(a; Y) \leq F(b; Z) \leq vF(a; Z_{n-1}) + (1-v)f(z_n), \tag{4.17}$$

where the vector a is defined in (4.5). Also,

$$F\left(\beta_n, b_n; \sum_{l=1}^{n-1} \rho_l z_l, z_n\right) \leq F(b; Z) \leq \sum_{l=1}^{n-1} \rho_l F(\beta_n, b_n; z_l, z_n). \tag{4.18}$$

Proof. Inequality (4.17) follows by use of [19, Eq. (5.7)] on [19, Eq. (3.8)]. Similarly, (4.18) can be obtained applying [19, Eq. (5.8)] to [19, Eq. (3.8)]. We omit further details. ▮

COROLLARY. *Let $b \in \mathbb{R}^n_>$, $X \in \mathbb{R}^n_>$. Then*

$$\mathcal{L}(a; x_1^v x_n^{1-v}, \ldots, x_{n-1}^v x_n^{1-v}) \leq \mathcal{L}(b; X) \leq v\mathcal{L}(a; x_1, \ldots, x_{n-1}) + (1-v)x_n \tag{4.19}$$

and

$$\mathcal{L}(\beta_n, b_n; \prod_{l=1}^{n} x_l^{\rho_l}, x_n) \leq \mathcal{L}(b; X) \leq \sum_{l=1}^{n-1} \rho_l \mathcal{L}(\beta_n, b_n; x_l, x_n), \tag{4.20}$$

where the vector a and the weights v and $\rho_1, \ldots, \rho_{n-1}$ are defined in (4.5) *and* (4.14), respectively.

Proof. In order to establish (4.19) we substitute $f(z) = \exp z$ and $Z = \ln X$ into (4.17). Use of (4.3) on the resulting inequality completes the proof. Inequality (4.20) can be established by the same means. ▮

To this end we will assume that the b-parameters of $\mathcal{L}(b; X)$ are positive integers. Our goal is to establish the following formula for the mean in question

$$\mathcal{L}(b; X) = (k-1)!\,[z_1, \ldots, z_k]e^t, \tag{4.21}$$

where $k = b_1 + \cdots + b_n$, the symbol $[z_1, \ldots, z_k]e^t$ stands for the divided difference of order $k - 1$ of $\exp(t)$ with knots at $z_1, \ldots, z_k$, and

$$z_i = \begin{cases} \ln x_1, & i = 1, \ldots, b_1, \\ \ln x_2, & i = b_1 + 1, \ldots, b_1 + b_2, \\ \vdots & \\ \ln x_n, & i = b_1 + \cdots + b_{n-1} + 1, \ldots, k. \end{cases}$$

For the proof of (4.21) we apply Property (iii) to (4.2) to obtain

$$\mathscr{L}(b; X) = S(d; \ln x_1(b_1), \ldots, \ln x_n(b_n)),$$

where $d = (1, \ldots, 1) \in \mathbb{R}^k$ and the symbol $\ln x_i(b_i)$ means that the variable $\ln x_i$ is repeated b_i-times. Combining this with (4.1) and (1.5) yields

$$\mathscr{L}(b; X) = (k - 1)! \int_{E_{k-1}} \exp\left(\sum_{i=1}^{k} u_i z_i\right) du. \tag{4.22}$$

The assertion now follows by use of the Hermite–Genocchi formula [2, Theorem 3.3]

$$[y_1, \ldots, y_k]f = \int_{E_{k-1}} f^{(k-1)}\left(\sum_{i=1}^{k} u_i y_i\right) du,$$

with $y_i = z_i$, $1 \leq i \leq k$, and $f(t) = \exp(t)$, on the right side of (4.22).

Using a well-known recurrence relation for the divided differences and the fact that

$$[\underbrace{\ln x, \ldots, \ln x}_{j\text{-times}}]e^t = x/(j - 1)!$$

one can design an algorithm for computing the mean under discussion. We omit further details.

If $b = (1, \ldots, 1) \in \mathbb{R}^n$ and if $x_i \neq x_j$ for all $i \neq j$, then the application of [2, Eq. (3.2.5)] to (4.21) leads to a closed formula

$$\mathscr{L}(b; X) = (n - 1)! \sum_{i=1}^{n} \left[x_i \Big/ \prod_{\substack{j=1 \\ j \neq i}}^{n} \ln(x_i/x_j) \right].$$

REFERENCES

1. W. H. ADAMS, "Heat Transmission," McGraw-Hill, New York, 1954.
2. K. E. ATKINSON, "An Introduction to Numerical Analysis," 2nd ed. Wiley, New York, 1989.
3. J. BRENNER AND B. C. CARLSON, Homogeneous mean values: Weights and asymptotics, *J. Math. Anal. Appl.* **123** (1987), 265–280.
4. P. S. BULLEN, D. S. MITRINOVIĆ, AND P. M. VASIĆ, Eds., "Means and Their Inequalities," Reidel, Dordrecht, 1989.
5. B. C. CARLSON, A hypergeometric mean value, *Proc. Amer. Math. Soc.* **16** (1965), 759–766.
6. B. C. CARLSON, Some inequalities for hypergeometric functions, *Proc. Amer. Math. Soc.* **17**(1966), 32–39.
7. B. C. CARLSON, The logarithmic mean, *Amer. Math. Monthly* **79** (1972), 615–618.
8. B. C. CARLSON, "Special Functions of Applied Mathematics," Academic Press, New York, 1977.
9. B. C. CARLSON AND M. D. TOBEY, A property of the hypergeometric mean value, *Proc. Amer. Math. Soc.* **19** (1968), 255–262.
10. H. B. CURRY AND I. J. SCHOENBERG, On Pólya frequency functions. IV. The fundamental spline functions and their limits, *J. Analyse Math.* **17** (1966), 71–107.
11. W. DAHMEN AND C. A. MICCHELLI, Recent progress in multivariate splines, *in* "Approximation Theory IV" (C. K. Chui, L. L. Schumaker, and J. D. Ward, Eds.), pp. 27–121, Academic Press, New York, 1983.
12. W. DAHMEN AND C. A. MICCHELLI, Statistical encounters with B-splines, *Contemp. Math.* **59,** (1986), 17–48.
13. A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, "Higher Transcendental Functions," McGraw-Hill, New York, 1953.
14. G. HÄMMERLIN AND K. H. HOFFMANN, "Numerical Mathematics," Springer-Verlag, New York, 1991.
15. T.-P. LIN, The power mean and the logarithmic mean, *Amer. Math. Monthly* **81** (1974), 879–883.
16. E. NEUMAN, Moments and Fourier transform of B-splines, *J. Comput. Appl. Math.* **7** (1981), 51–62.
17. E. NEUMAN, On complete symmetric functions, *SIAM J. Math. Anal.* **19** (1988), 736–750.
18. E. NEUMAN, Inequalities involving multivariate convex functions, II, *Proc. Amer. Math. Soc.* **109** (1990), 965–974.
19. E. NEUMAN, Dirichlet averages and their applications to special functions, submitted for publication.
20. E. NEUMAN AND P. VAN FLEET, Moments of Dirichlet splines and their applications to hypergeometric functions, J. Comput. Appl. Math, in press.
21. A. O. PITTENGER, The symmetric, logarithmic, and power means, Univ. Beograd, Publ. Elektrotehn. Fak., Ser. Mat. Fiz., No. 681 (1980), 19–23.
22. A. O. PITTENGER, The logarithmic mean in n variables, *Amer. Math. Monthly* **92** (1985), 99–104.
23. A. O. PITTENGER, Limit theorems for logarithmic means, *J. Math. Anal. Appl.* **12** (1987), 282–291.
24. G. PÓLYA AND G. SZEGÖ, Isoperimetric Inequalities in Mathematical Physics," Princeton Univ. Press, Princeton, NJ, 1951.